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2.8   2.5

3.2   2.2

2.0

1.1

1.8

1.25   1.4   1.6

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A new regime is found in which the growth rate of the resistive interchange mode in cylindrical geometry is considerably smaller than previously believed. This stabilization is due to coupling to the external, tearing stable, region. Considering the crossover point from the conventional resistive interchange to the new regime as a critical beta, β_c, we find $\beta_c \sim 7\%$ for classical resistivity with $T_e = 10$ eV, in agreement with recent numerical studies. However, β_c scales as $T_e^{-3/5}$, giving $\beta_c \sim 1\%$ for $T_e = 1$ keV. Marginally stable pressure profiles for totally localized modes in the Bessel function model for a reversed field pinch are also computed, and show that the central pressure may be up to 400 times the pressure at the wall.

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RESISTIVE INTERCHANGE MODES IN
REVERSED FIELD PINCHES

I. Introduction

It has long been recognized that reversed field pinches, having unfavorable average curvature should be unstable to resistive interchange modes for arbitrarily small β and for arbitrarily large m and n (poloidal and toroidal mode numbers) unless finite Larmor radius effects are taken into account.[1,2] The theory of gravity driven modes in an incompressible fluid in a sheared field indicates $\gamma \sim \eta^{1/3} |p'/q'|^{2/3}$ where γ is the growth rate, η the resistivity, p' the pressure gradient and $q = \dfrac{rB_z}{RB_\theta}$. However recent numerical results pertaining to the linear and nonlinear behavior of these modes by Schnack et al.[3] indicate that at low β, the plasma is in fact much more stable than this result would indicate. We find that a careful analytic study of the theory of resistive interchange modes show that there are several very important stabilizing effects which are not included in the simple slab model with gravity simulating curvature and have been ignored in previous treatments in cylindrical geometry.

Most resistive interchange modes are localized to within the singular layer. For these modes, the effect of compressibility and shear turn out to be strong stabilizing effects. If the plasma compression term, proportional to Γp (an upper case Γ is used to avoid confusion with the growth rate γ) is not negligible, the plasma can support a pressure gradient and be interchange stable even in the absence of shear. This was first pointed out by Bernstein et al.[4] for a plasma with $B_z = 0$. Also, for $B_z \neq 0$ there is some residual shear stabilization which can further stabilize interchange modes in a resistive plasma. Thus, most of the resistive interchange modes can be stabilized by this combination of compressibility and shear. The marginal stability condition gives rise to a marginally stable relative pressure profile in a reversed field pinch.

However, if these modes are stable, there is one additional unstable mode which couples to the plasma outside of the singular layer. This instability exists for any negative pressure gradients. Above a critical pressure gradient, the growth rate scales as $\gamma \sim (-p')^{2/3}$, like a conventional resistive interchange mode. Below the critical pressure gradient however, $\gamma \sim (-p')^4$, so that the growth rate goes to zero extremely quickly as $p' \to 0$. Furthermore γ scales as $|\Delta'|^{-4}$, where Δ' is proportional to the jump in the radial derivative of the perturbed radial magnetic field, so that short wavelength modes are much less dangerous than the longest wavelength ($m = 0$ and $m = 1$) modes. For all practical purposes, this critical pressure gradient then defines a marginal stability pressure for this last resistive interchange mode.

Section II reviews the equations and scalings for these modes, as originally derived by Coppi, Greene & Johnson (CGJ).[5] Section III derives the properties of the localized modes and calculates relative pressure profiles for a reversed field pinch, using the Bessel function equilibrium model. We conclude that the marginally stable pressure profile can have a central pressure up to 400 times the pressure at the walls.

In Section IV we discuss the properties of the mode that couples to the fluid outside the singular layer. In several limits (high shear, low shear), assuming parameters so that the previously discussed localized modes are stable, the dispersion relation takes on a very simple form, showing the $\gamma \sim (-p')^4$ scaling as $p' \to 0$. We also show that the dispersion relation can be obtained very simply in the low shear, low pressure gradient limit. We evaluate Δ' for modes with various toroidal and poloidal mode numbers n,m, using the Bessel function model again and show that, if the pitch parameter is large enough, the $m = 0$ modes are the most dangerous. This result was observed in simulations by Schnack et al.[3] We also evaluate the critical β for reversed field pinches. The critical β turns out to be about 7 percent for a magnetic Reynolds number $R_m = 10^3$ ($T_e \sim 10$ eV) but less than one percent for kilovolt temperatures ($R_m > 10^5$). The former result agrees with the numerical results of Schnack et al.[3] The critical beta decreases with conductivity because the stabilizing coupling to the outer (tearing stable) region becomes weaker with increasing conductivity.

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II. Review of the Equations & Scalings

In this section we briefly review the Equations and scaling as derived in CGJ. Outside a small distance L_r away from the rational surface, the ideal MHD equations hold, but within this distance L_r, finite resistivity is important. As we will see, there are some modes which are localized within L_r and which do not couple at all to the outer region. For other modes, there is strong coupling to the outer region. We will assume that $L_r/a \sim \epsilon$ where a is the radius of the mode rational surface and a is also assumed to be roughly equal to the scale of variation of the eigenfunction in the other direction perpendicular to $\underline{B}$. Since the plasma is assumed to be ideal MHD stable to localized modes (Suydam stable), the growth rate also scales as ϵ. In this, the so-called glow interchange ordering of CGJ, ϵ also scales as $\eta^{1/3}$.

In cylindrical geometry, the eigenfunction has the form $\tilde{f}(r) \exp(\gamma t + im\theta + ikz)$ where in a large aspect ratio torus $k = -n/R$. A superscript $\sim$ denotes a perturbed quantity. The mode rational surface is determined by

$$m = nq \tag{1}$$

where $q = rB_z/RB_\theta$. The operator $\underline{B} \cdot \underline{\nabla}$ operating on a scalar is given by

$$B \cdot \nabla \rightarrow -\frac{in\,B_\theta\,q'}{a}(r-a) + 0(r-a)^2 \tag{2}$$

and within the singular region is of order ϵ. (We use this form rather than that of CGJ since it applies also to $m = 0$ modes, which have their rational surface where $B_z = 0$.)

Since the resistive interchange mode is characterized by nonzero resistivity allowing the fluid to slip through the sheared field near the rational surface, the mode is mostly fluid flow with only a small magnetic perturbation. Thus $\underline{\tilde{B}} \sim \epsilon \tilde{\xi}$ where $\tilde{\xi}$ is the linearized fluid displacement. Since the growth rate is assumed to be slow compared to the magnetosonic speed, we have

$$B \cdot \tilde{B} + \tilde{p} = 0. \tag{3}$$

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Since $\tilde{p} = -p'\xi_r - \Gamma p (\nabla \cdot \xi)$, where Γ is the specific heat ratio, both ξ_r and $(\nabla \cdot \xi)$ must be of order ϵ. Thus to zero order the flow is divergence free. However the small compression does play an important role in the pressure perturbation. Since $\tilde{B}$ is also divergence free, $\tilde{B}_r \sim \epsilon \ \tilde{B}_{\theta,z}$.

If we define a local coordinate system $\underline{i}_r$, $\underline{i}_\parallel$ and $\underline{i}_\perp = \underline{i}_r \times \underline{i}_\parallel$, where $\underline{i}_r + \underline{i}_\parallel$ are unit vectors in the directions of $\underline{r}$ and $\underline{B}$, then

$$\tilde{\xi} = \tilde{\xi}_r \ \underline{i}_r + \tilde{\xi}_\parallel \ \underline{i}_\parallel + \tilde{\xi}_\perp \ \underline{i}_\perp \tag{4}$$

$$\tilde{\underline{B}} = \tilde{B}_r \underline{i}_r + \tilde{B}_\parallel \ \underline{i}_\parallel + \tilde{B}_\perp \ \underline{i}_\perp \tag{5}$$

where $\tilde{\xi}_{\parallel,\perp} \sim \epsilon^0$, $\tilde{\xi}_r$, $\tilde{B}_{\perp,\parallel} \sim \epsilon$ and $\tilde{B}_r \sim \epsilon^2$. The equations for $\tilde{\xi}$, $\tilde{B}$, and $\nabla \cdot \tilde{\xi}$ to lowest order are given by CGJ as

$$\tilde{\xi}_\perp + i \ \frac{RB_\theta}{n B} \ \frac{d\tilde{\xi}_r}{dx} = 0 \tag{6}$$

$$\tilde{B}_\perp + i \ \frac{RB_\theta}{n B} \ \frac{d \tilde{B}_r}{dx} = 0 \tag{7}$$

$$B \tilde{B}_\parallel = p'\tilde{\xi}_r + \Gamma p (\nabla \cdot \tilde{\xi}) \tag{8}$$

$$\rho \gamma^2 B\tilde{\xi}_\parallel = - p' \tilde{B}_r - \frac{i n B_\theta q' Bx}{a} \ \tilde{B}_\parallel \tag{9}$$

$$\rho \gamma^2 \tilde{\xi}''_r = \frac{2 n^2 B}{R^2 a} \ \tilde{B}_\parallel - \frac{i n B_\theta q'x}{a} \ \tilde{B}''_r \tag{10}$$

$$\tilde{B}_r - \frac{\eta}{\gamma} \ \tilde{B}''_r = - \frac{i n B_\theta q'x}{a}\tilde{\xi}_r \tag{11}$$

$$\tilde{B}_\parallel - \frac{\eta}{\gamma} \ \tilde{B}''_\parallel = - \frac{i n B_\theta q'x}{a} \tilde{\xi}_\parallel - B (\nabla \cdot \tilde{\xi}) \tag{12}$$

$$\frac{- (p + B^2)'}{B} \ \tilde{\xi}_r$$

where $x = r - a$. Equations (6)- (12) are CGJ eqs. (47) - (52) with several slight changes in notation: we have used q instead of the rotational transform since q is a nonsingular function of r for a reversed field pinch; we have used γ for growth rate and Γ for specific heat ratio; finally all components of $\tilde{B}$ and $\tilde{\xi}$ have dimension magnetic field and length respectively.

Equations (6) and (7) set the ϵ^0 part of $\nabla \cdot \tilde{\xi}$ and $\nabla \cdot \tilde{B}$ equal to zero. Equation (8) sets the perturbed magnetic plus thermal pressure equal to zero. Equation (9) is the dot product of $\underline{B}$ with the momentum equation. Eq. (10) results from operating on the momentum equation with the operator

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$\nabla \cdot (\underline{B}/B^2 x)$ Coppi, Greene and Johnson[5] call this the annihilated momentum equation since the operator annihilated information about propagation along the lines of force (given by eq. (9)) and also annihilates information about $\nabla (B \cdot \tilde{B} + \tilde{p})$ (given by eq. (3)). Equations (11) and (12) are the radial and $\underline{i}_\parallel$ components of Ohm's law.

The first two terms of (12) combine to give $-B (\nabla_\perp \cdot \xi_\perp)$, the compression of the field due to the compression of the fluid perpendicular to $\underline{B}$. The last term is a combination of two effects. First there is the convection of $\underline{B}$ into regions of different B. This gives a contribution to $\tilde{B}_\parallel$ of

$$\delta B_\parallel = \frac{-B \cdot (\tilde{\xi} \cdot \nabla B)}{B} = -\tilde{\xi}_r \frac{\partial}{\partial r} \left[\frac{B^2}{2}\right]. \tag{13}$$

Secondly, consider a flux tube at radius in which is incompressibly displaced a distance $\tilde{\xi}_r$. If the tube originally has area A, the change in area of the displaced tube is $\delta A = -\frac{A}{r} \tilde{\xi}_r$. However since the flux is conserved, then

$$\delta B_\parallel = -\frac{B}{A} \delta A = \frac{B}{r} \tilde{\xi}_r, \tag{14}$$

$$= -\frac{\tilde{\xi}_r}{B} \frac{\partial}{\partial r} \left[p + \frac{B^2}{2}\right]. \tag{15}$$

The last relation follows from pressure balance. Combining Eqs. (13) and (15) gives the last term in Eq. (12). The term on the right of (11) is the analog of the first term on the right in (12) or, alternatively, the convection of the helical flux $\psi \equiv i r \tilde{B}_r$.

Equations (8) and (9) give $\nabla \cdot \tilde{\xi}$ and $\tilde{\xi}_\parallel$ in terms of the other variables. In the notation of CGJ, the equations are

$$\Psi'' = Q (\Psi + X \Xi) \tag{16}$$

$$\Xi'' = \frac{X^2}{Q} \Xi + \frac{X}{Q} \Psi - \frac{1}{Q^2} Y \tag{17}$$

$$Y'' = Q\left[1 + \frac{X^2}{Q^2} + \frac{2}{\Gamma \beta}\right] Y + Q\left[S - D - \frac{2D}{\Gamma \beta}\right] \Xi + \frac{D}{Q} X \Psi \tag{18}$$

where $Q = \gamma/Q_r$, $\Xi = \xi_r$, $\Psi \equiv \left[\frac{-i a}{n B_\theta q' L_r}\right] \tilde{B}_r$, $X = x/L_r$, $Y = \left[\frac{-2 B a}{R^2 q'^2 B_\theta^2}\right] \tilde{B}_\parallel$, and

$$D = -\frac{2p'a}{B_\theta^2 R^2 q'^2} \tag{19}$$

$$\beta = 2p(a)/B^2 \tag{20}$$

$$S = 4/R^2 q'^2 \tag{21}$$

$$L_r = \left[\frac{\rho \eta^2 a^2}{n^2 B_\theta^2 q'^2}\right]^{1/6} \tag{22}$$

$$Q_r = \left[\frac{\eta \, n^2 B_\theta^2 q'^2}{\rho \, a^2}\right]^{1/3} . \tag{23}$$

We have eliminated m, B_z and $\iota = 2\pi/q$ because in a reversed field pinch the mode rational surface for $m = 0$ modes has $B_z = 0$.

As long as there is no coupling to the outer region, the solutions to Eqs. (16) - (18) have either odd or even symmetry, with the symmetry of Ψ opposite to the symmetry of Ξ and Y.

As shown in CGJ, near marginal stability these modes are localized within a more narrow layer, with $X \sim Q^{1/4}$, and therefore modes with odd Ξ, even Ψ and Y in fact must be treated by tearing ordering. The most important effects of this ordering, where $\gamma \sim \eta^{3/5}$, $r - r_s \sim \eta^{2/5}$ and $\beta \sim \eta^{2/5}$, is that the first term on the right in (16) can be treated to lowest order as a constant. (This is the constant-ψ approximation of Ref. 6.) Also, the first term in the Y coefficient in (18) and the second term in the Ξ coefficient are negligible.

III. Even Ξ Modes and Stable Pressure Profiles

In this section we review the even Ξ modes.[5,6] For modes of this symmetry it is consistent with the constant Ψ approximation to neglect the first term of (11) i.e., the first term on the right in (16). Therefore, Eqs. (17) and (18) decouple from Eq. (16) for Ψ. With tearing ordering (i.e., $\beta \sim \eta^{2/5}$), Eqs. (17) and (18) become

$$\Xi'' = \frac{X^2}{Q} \Xi - \frac{Y}{Q^2} \tag{24}$$

$$Y'' = Q(\frac{X^2}{Q^2} + \frac{2}{\Gamma\beta}) Y + Q(S - \frac{2D}{\Gamma\beta})\Xi. \tag{25}$$

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Following CGJ we expand in Hermite functions $\Xi = \sum_l \Xi_l \, H_l(z) \, \exp(-z^2/2)$, $Y = \sum_l Y_l \, H_l(z)$

$\exp(-z^2/2)$, where $z = X/Q^{1/4}$.

Making use of the fact that

$$\left[\frac{d^2}{dz^2} - z^2\right] H_l(z) \, \exp\left(-\frac{z^2}{2}\right) = -(2l+1) \, H_l(z) \, \exp(-z^2/2)$$

one finds for even l two simultaneous equations for Ξ_l and Y_l,

$$-(2l+1) \, \Xi_l + Q^{-3/2} Y_l = 0$$

$$-Q^{3/2}\left[S - \frac{2D}{\Gamma\beta}\right] \Xi_l + \left[-(2l+1) - \frac{2Q^{3/2}}{\Gamma\beta}\right] Y_l = 0 \qquad (26)$$

which give the dispersion relation

$$Q^{3/2} = \frac{\Gamma\beta}{2}\left[\frac{\dfrac{2D}{\Gamma\beta} - S}{2l+1} - (2l+1)\right] \qquad (27)$$

The most unstable mode has $l = 0$; therefore stability to all even Ξ modes is guaranteed if

$$\frac{2D}{\Gamma\beta} < S + 1 \qquad (28a)$$

or in physical units

$$-\frac{2p'a}{R^2 B_\theta^2 q'^2} < \frac{\Gamma p}{B^2}\left[\frac{4}{R^2 q'^2} + 1\right]. \qquad (28b)$$

The even $\tilde{\xi}_r$ modes have $\tilde{\xi}_r$ and $\tilde{B}_{\parallel}$ entirely localized to a region $X < Q^{1/4}$ inside the singular layer and do not couple to the plasma in the outer region. To check the validity of the constant Ψ approximation we need only show a postiori that $Q\psi << \psi''$. Using $\Psi'' \sim \Psi/Q^{1/2}$, we find the condition for validity of the constant ψ approximation is

$$Q^{3/2} \sim D \sim \beta << 1 \qquad (29)$$

Notice that unlike a gravity driven resistive interchange in slab geometry[6], the even Ξ modes in a cylinder are stable for a pressure gradient below a critical value given by Eq. (28b).

This critical pressure gradient can be related to previous work[4] on the ideal MHD stability of a compressible diffuse pinch with $B_z = 0$. According to Bernstein et al.[4], a diffuse z pinch, which has unfavorable curvature everywhere, is stable to a pure interchange mode only if

$$\frac{V''}{V'} + \frac{p'}{\Gamma p} > 0 \tag{30}$$

where V is proportional to the flux tube volume r/B_θ. Thus, increasing Γp (i.e., compressibility) has a stabilizing effect. It is not difficult to show that (30) is proportional to the factor $S - D(2/\Gamma\beta + 1)$ of Eq. (18) if $B_z = 0$ is assumed. The third term in this factor is necessary because β must be of order unity if $B_z = 0$.

Increasing B_z has both stabilizing and destabilizing effects on even Ξ modes. The destablizing effect, given by the factor $\Gamma p/B^2$ in (28b), is due to the fact that large B_z implies incompressibility (but without $C_s^2 = \Gamma p/\rho \rightarrow \infty$) so that no work is done in compressing the plasma. The stabilizing effect of increasing B_z is that this may increase the shear [represented by the second term on the right of Eq. (28a) or Eq. (28b)].

To get an idea of the stable pressure profiles in a reversed field pinch, we have computed the marginal pressure profile from Eq. (28b) for a Bessel function model

$$B_z = B_0 J_0(\mu r) \tag{31}$$
$$B_\theta = B_0 J_1(\mu r).$$

This is the magnetic field produced by the plasma if it relaxes to minimum energy state while maintaining constant helicity[7], giving $\nabla \times \underline{B} = \mu \underline{B}$. Although the field given by Eq. (31) is force free, it should still be a good approximation to the field structure in a low β plasma. If the quantity r_w is the wall radius, the range of μ of most interest to a reversed field pinch is

$$2.4 < \mu r_w < 3.1,$$

the former value being the value required for field reversal, the latter required for tearing mode stability.[8]

Since Eq. (28b) gives p'/p it gives only relative, not absolute, pressure profiles. Thus, if the boundary condition is $p(r_w) = 0$, there must be an unstable region somewhere in the plasma. In Fig. (1) are shown plots of relative pressure profiles for five different values of μ. Clearly as μ increases, the pressure at the center can become quite large compared to the pressure at the walls. For instance at $\mu = 3.1$, just below the threshold for tearing mode instability, the pressure at the center is more than 400 times the pressure at the wall. It is of interest to note that the toroidal factor H of Ref. 9 [eq. (13)] is zero for toroidal equilibria which satisfy $\nabla \times \underline{B} = \mu \, \underline{B}$ with $\mu = \underline{j} \cdot \underline{B}/\underline{B}^2$ constant.

IV. Odd Ξ Modes—The Maximum Stable Pressure

In the previous section we have seen that if the pressure gradient is smaller than that given by Eq. (28a or b), then the even Ξ modes are stable. One might also think odd Ξ modes stable also since they have larger l. However, since Ξ has odd symmetry, Ψ has even symmetry, so it can couple to the outer MHD stable region through the (constant) value of Ψ. This allows for the presence of one unstable mode even if (b) is violated.

The inner region equations for odd Ξ modes are

$$\Xi'' - z^2\Xi + Q^{-3/2}Y = Q^{-1/4}z\Psi_0 \tag{32}$$

$$Y'' - z^2Y - Q^{3/2}(S - 2D/\Gamma\beta)\,\Xi$$
$$- 2Q^{3/2}Y/\Gamma\beta = DQ^{-1/4}z\Psi_0. \tag{33}$$

Henceforth, we assume that the even Ξ modes are stable, i.e., $S - 2D/\Gamma\beta > -1$. Earlier references[5,6] have shown that this equation can be analyzed by expanding in Hermite functions and expressing the resulting series as a hypergeometric series. In the incompressible limit $2D/\Gamma\beta \to 0$, we find, as in Refs. 5, 9,

$$L_r\Delta' = \frac{4\pi^2}{\sqrt{2}\,|\Gamma(1/4)|^2}\,(Q^{5/4} - \pi D/4Q^{1/4}), \tag{34}$$

where $\Delta' = [\partial\Psi/\partial r\,(a+) - \partial\Psi/\partial r\,(a-)]/\Psi(a)$. In the limit $S \gg 1$ (in which $2D/\Gamma\beta$ need not be small, and which corresponds to the low shear limit) we find

$$L, \Delta' = \frac{\pi}{\sqrt{2}} \left(Q^{5/4} \sigma^{1/4} - D/Q^{1/4} \sigma^{1/4} \right). \tag{35}$$

where $\sigma = S - 2D/\Gamma\beta$.

In the Appendix we show that it is possible to find a simple closed-form solution to (32), (33) in this latter limit. In fact, we find

$$\Xi(z) = - \frac{\Psi_0 Q^{-1/4} z (D/Q^{3/2} + z^2)}{z^4 + \sigma}. \tag{36}$$

We are also able to obtain (35) exactly.

It has been shown by using Nyquist analysis that equations of the form (34) or (35) always have one root in the right half Q plane.[9] A simple graphical analysis shows that this root is pure real. If $\Delta' > 0$, $D \ll 1$, the Δ' balances the $Q^{5/4}$ term on the right hand side of eq. (34) to give the conventional tearing mode dispersion relation. On the other hand, if $\Delta' = 0$, one easily calculates from (34) $Q = (\pi D/4)^{2/3}$, the conventional result for a resistive interchange mode. Using (35) we obtain a similar result $Q = D^{2/3}/\sigma^{1/3}$. The most interesting limit, however, is for Δ' negative and either $|\Delta'| \gg 1$ or $D \ll 1$. In this case (34) gives

$$Q \simeq 7.7 \, (D/L, \Delta')^4. \tag{37}$$

This growth rate goes to zero very rapidly as $D \to 0$. The result complements that of Glasser, Greene & Johnson[9], who show that coupling to favorable average curvature can stabilize a tearing mode. Here we show the converse, that coupling to large, negative Δ' can substantially reduce the growth rate of a resistive interchange mode. The growth rate goes to zero so rapidly with D that we can consider the transition point from $D^{2/3}$ to D^4 behavior, namely $D_c = .52 \, (- L, \Delta')^{6/5}$, to be an effective critical beta for the mode. Using (35) the corresponding results are $Q \simeq (24/\sigma) \, (D/L, \Delta')^4$ and $D_c = .38 \sigma^{1/5} (- L, \Delta')^{6/5}$. Note that (37) gives $\gamma \sim p'^4 \eta \, q'^6$, whereas the $\sigma \to \infty$ result gives $\gamma \sim p'^4/\eta \, q'^4$. In both cases $\gamma \sim \eta^{3/5}$.

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In physical variables the critical pressure gradient obtained from D_c is

$$- dp/dr = (B_\theta^2 R^2 q'^2/4) (- \Delta' L_r)^{6/5} \tag{38a}$$

$$= (B_\theta^2 R^2 q'^2/4) (- \Delta')^{6/5} \left[\frac{\rho \eta^2 a^2}{n^2 q'^2 B_\theta^2} \right]^{1/5}. \tag{38b}$$

Note that, for plasma near the critical β for low m,n modes, modes with larger m,n have a much smaller growth rate since Q scales as $(\Delta')^{-4}$ and since $\Delta' \sim - n$ for $n \to \infty$. In particular, since $Q_r \sim n^{2/3}$, we have $\gamma \sim n^{-10/3}$ as $n \to \infty$. Another way of expressing this is to note that the critical pressure gradient scales as $n^{4/5}$ for large n. Note that the critical pressure gradient scales as $\eta^{2/5}$. This is as expected, since it is tearing ordering with $\beta \sim \eta^{2/5}$ that makes the three terms of (34) or of (35) comparable.

For the force free Bessel function model $B_\theta = B_0 J_1 (\mu r)$, $B_z = B_0 J_0 (\mu r)$, we can easily compute Δ' analytically. This is possible because the outer region equations for this equilibrium are merely $\nabla \times \tilde{B} = \mu \tilde{B}$, with the same value of μ as the equilibrium. (Recall from Sec. III that toroidal generalizations of such equilibria have the toroidal term H of Ref. 9 equal to zero.) In this model field reversal occurs if $\mu r_w > 2.4$ (r_w is the plasma minor radius) and an $m = 1$ tearing mode is unstable if $\mu r_w > 3.1$. In Fig. 2a we show the maximum value of $\Delta' r_w$ (over $0 \leqslant m \leqslant 5, -25 \leqslant n \leqslant 25$) as a function of r for $\mu r_w = 2.81$. Fig. 2b shows the maximum of $\Delta' r_w$ over n for various m, as a function of μr_w. These results show that for $2.4 < \mu r_w < 2.83$, the $m = 1$ mode with $n > 0$ (i.e., with mode rational surface inside the field reversal point) has the largest Δ'. The $m = 1$ mode with largest Δ' has $n r_w/R \simeq 2$, which has the effect of putting the mode rational surface as far from the walls and from the origin $r = 0$ as possible. For $2.83 < \mu r_w < 3.1$, however, the $m = 0$ mode (which has its mode rational surface at the field null $\mu a = 2.4$ regardless of n) with $n = \pm 1$ has the largest value of Δ'. We also observe that an $m = 1$ tearing mode with $n < 0$ (i.e., with mode rational surface outside the field null) becomes unstable (i.e., $\Delta' > 0$) for $\mu r_w > 3.1$, in agreement with previous studies.[1,8]

Putting $\Delta' = -4$, we find the critical D for $m = 0$ modes to be approximately $D_c = 3 R_m^{-2/5}$, where R_m is the magnetic Reynolds number. For $R_m = 10^3$, as in Ref. 3, we find $D_c \sim 20$ percent. For $R_m = 10^5$, we find $D_c \simeq 3$ percent. For the marginal equilibria of Sec. III, we have $\beta \sim D/3$, giving

critical β equal to 7 and one percent, respectively. The former result is in agreement with the simulations of Schnack et al.[3]

In summary, we find that matching to a tearing stable exterior region (i.e., with $\Delta' < 0$) decreases growth rates for the resistive interchange sufficiently that below a critical value of the Suydam parameter D, the growth rate scales as D^4 and is therefore quite small. We also find that in this regime, the growth rate γ scales as $|\Delta'|^{-4}$, so that for very short wavelength modes ($n \to \infty$), γ scales as $n^{-10/3}$. From a calculation of Δ' from the outer region, we find that the most unstable mode is either an $m = 1$ mode with $n r_w/R \simeq 2$ or an $m = 0$ mode with $n = \pm 1$. The above results, notably the abrupt dropoff in growth rate for small D, the decrease in growth rate for large n and the predominance of the $m = 0$, $m = 1$ modes, are in agreement with the numerical simulations of Schnack et al.[3]

ACKNOWLEDGMENTS

APPENDIX

ODD MODES FOR S >>1

In this Appendix we show that if $\sigma \equiv S - 2D/\Gamma\beta >> 1$, the eigenfunction in the inner region can be obtained analytically from (32), (33). We first note that in this limit, the fourth term on the left of (33) is negligible. (In fact this can be shown a posteriori). The resulting equations can be combined simply to give a single second order equation in terms of a complex variable.

$$Y = iQ^{3/2}\sigma^{1/2}\,\Xi + \Upsilon, \tag{A1}$$

namely

$$\Upsilon'' - z^2\,\Upsilon + i\sigma^{1/2}\,\Upsilon = \delta z, \tag{A2}$$

where

$$\delta = (iQ^{3/2}\sigma^{1/2} + D)\,\Psi_o\,Q^{-1/4}. \tag{A3}$$

For $\sigma >> 1$, solutions having $\Upsilon = \Upsilon(z/\sigma^{1/4})$ have the first term in (A2) of order σ^{-1} relative to the third term for all z. Therefore, to lowest order in σ^{-1} we have

$$\Upsilon = \Upsilon_0 = -\frac{\delta z}{z^2 - i\sigma^{1/2}}. \tag{A4}$$

Note that in this limit the radial component of inertia and the resistive term from (12) are small but the parallel component of inertia and resistive term from (11) are kept.

To next order we obtain the correction

$$\Upsilon_1 = -\frac{2\delta z(z^2 + 3i\sigma^{1/2})}{(z^2 - i\sigma^{1/2})^4}. \tag{A5}$$

Using the lowest order term (A4) we obtain, from (A1)

$$\Xi = Q^{-3/2}\sigma^{-1/2}\,\text{Im}\,\Upsilon = -\frac{z\Psi_0 Q^{-1/4}(DQ^{-3/2} + z^2)}{z^4 + \sigma} \tag{A6}$$

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From this we obtain

$$L_r \Delta' = \frac{Q^{5/4}}{\Psi_0} \int\limits_{\infty}^{\infty} dz \left[\Psi_0 + Q^{1/4} z \bar{\Xi}(z) \right] dz \qquad (A7)$$

or

$$L_r \Delta' = \frac{\pi}{\sqrt{2}} \left[\sigma^{1/4} Q^{5/4} - D/\sigma^{1/4} Q^{1/4} \right], \qquad (A8)$$

which is exactly (35). For $S >> 1$ and $2D/\Gamma\beta \sim 1$, $Q^{3/2}$ is bounded by $D/S^{1/2}$, so the third and fourth terms of (33) are in relation $S: Q^{3/2}/D$ [using (32)] or $S: S^{-1/2}$. This proves that the approximation discussed in the first paragraph of this appendix is valid.

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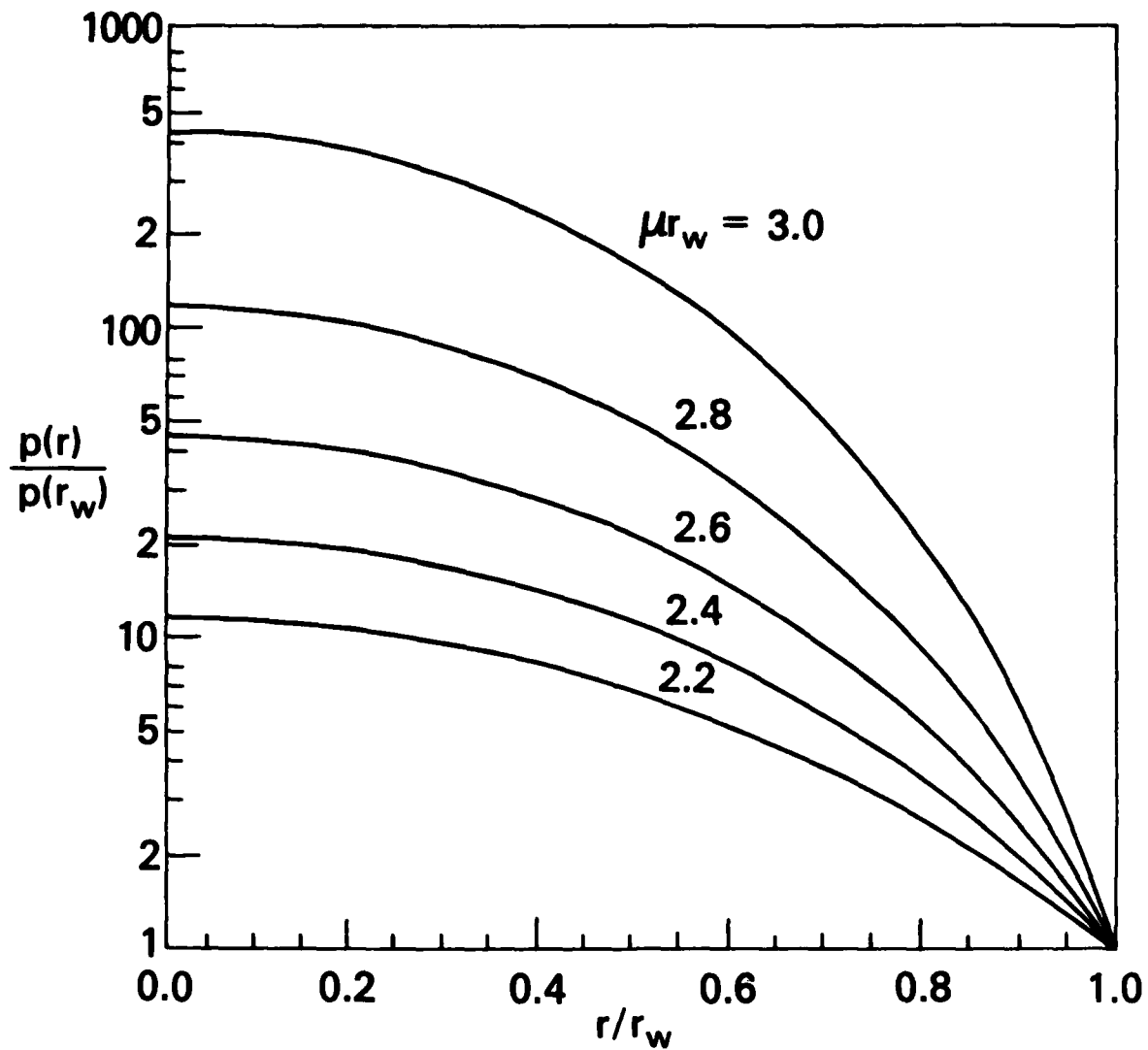
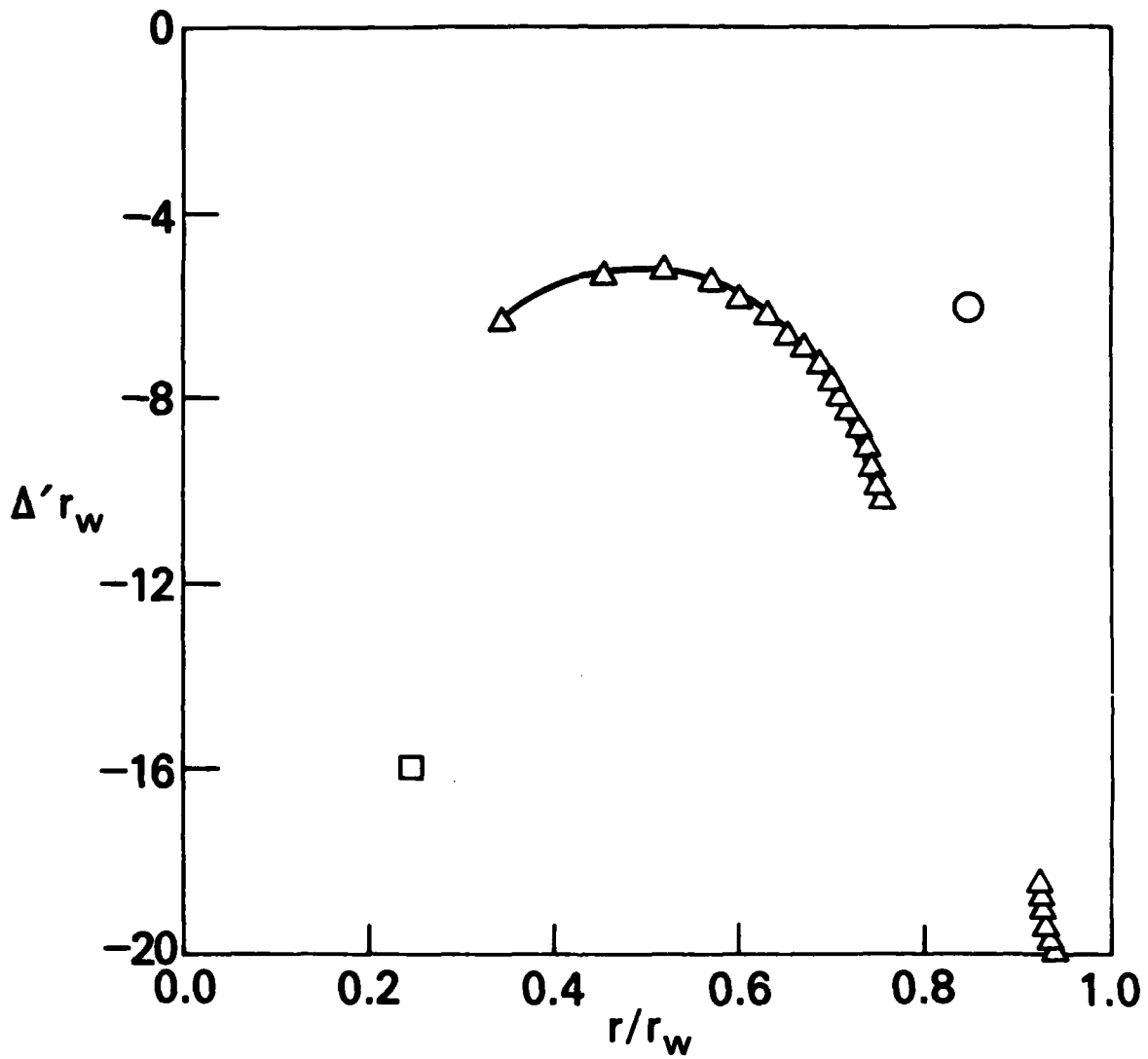Fig. 1 — Marginally stable relative pressure profiles (even Ξ modes) for the Bessel function equilibria

Fig. 2a — Maximum value of $\Delta' r_w$, over $0 \leq m \leq 5$, $-25 \leq n \leq 25$, as a function of r, for $\mu\, r_w = 2.81$. The symbols 0, Δ, and $\square$ refer to $m = 0$, $m = 1$ and $m = 2$ modes, respectively. The aspect ratio $R/r_w \equiv \epsilon^{-1}$ is 5. For ϵ smaller the spacing between mode rational surfaces would be smaller.
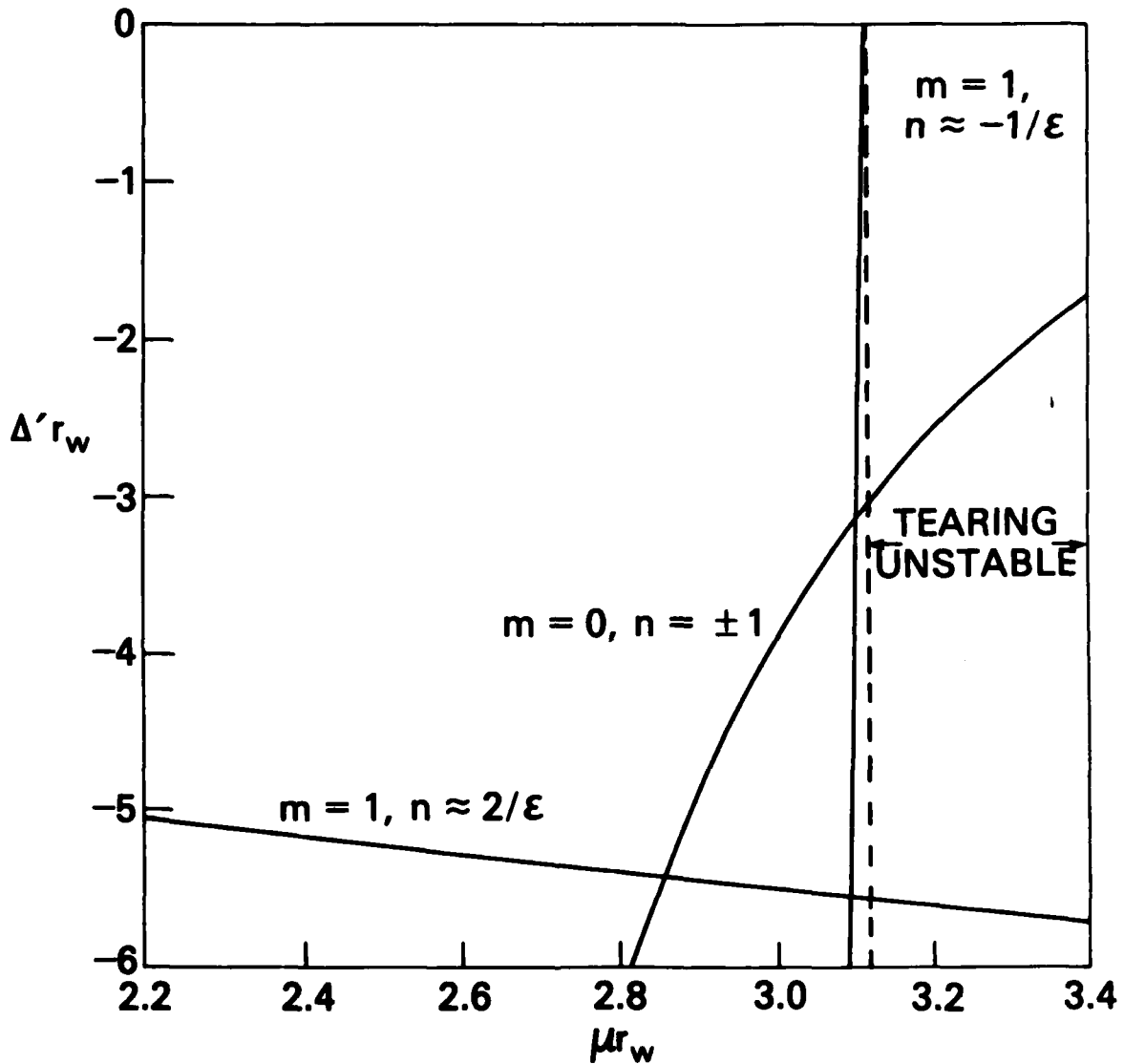
Fig. 2b — Maximum value of $\Delta' r_w$ over n for $m = 0$, $m = 1$ modes with $\epsilon \equiv r_w/R$. Notice that $m = 0$ modes have the larger Δ' for $\mu r_w > 2.83$

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